Stability of a Queuing System

This is a chapter from the dissertation: Z.I. Flikop "Structural Optimization of Operational Management System", Kiev Polytechnic Institute, Kiev, 1973. (In Russian).

In any queuing system we can define a P_{z^k} probability that z^k customers requiring type k service are simultaneously servicing by the system. We also can consider an expected value M_{z^k} that z^k customers are in the system simultaneously. It is obvious, that different queuing systems have different acceptable values of P_{z^k} and M_{z^k}, and, in course of functioning P_{z^k}, and M_{z^k} can exceed those values. This can be as a result, for example, changes in intensity of input customer traffic, productivity of servicing elements, queuing system configuration, and so on.

We should find a way to evaluate an effect of those changes on a queuing system performance. We can evaluate this system from point of view of its stability in average and its stochastic stability.

Let's start our analysis from studying queuing system stability in average. If considered system allows that

$$\frac{\lim}{t \to T \ \mathrm{p} \ \infty} M_{z^k}(t) \equiv M_{z^k} \ \mathrm{p} \ \infty \text{ for all } k, \qquad (1)$$

Then we can define a vector

$$M_z = (M_{z^1}, \ldots, M_{z^k}, \ldots, M_{z^d}) \qquad (2)$$

And a set Π values of M_z such

$$\Pi = \{M_{z^1} \le \mathrm{p}^1, \ldots, M_{z^k} \le \mathrm{p}^k, \ldots, M_{z^d} \le \mathrm{p}^d\} \ , \qquad (3)$$

where p^k is a maximum average accepted value of number of customers of type k that are servicing by system simultaneously.

In corresponding with [1, 2, 3] we will consider that after initial stabilization at time T_1:

1) queuing system is stable in average on interval $[T_1 T_2]$, if

$$M_z \in \Pi \quad \text{for} \quad t \in [T_1 T_2] \tag{4}$$

2) queuing system is stable in average on interval $[T_1 T_2]$ in relation to k type of customers, if

$$M_{z^k} \leq \text{p}^k \quad \text{for} \quad t \in [T_1 T_2] \tag{5}$$

We use time interval $t \in [T_1 T_2]$, because queuing system properties are chancing with time.

Condition (1) is satisfied when load of each server in the system is below one. However, in practice, we need satisfaction conditions (4) or (5) to be able to provide same excesses of system stability. This is needed to compensate for fluctuations in the input traffic intensity, and variations in the system configuration and the serving elements performance.

We can consider a queuing system that provides service to a pool of customers. This pool can contain as limited as unlimited number of customers. In case when pool is limited, number of customers in the pool depends on how many of them are in the servicing system, (customers after service are returning to the pool), i.e. intensity of customers input traffic varies and depends on number of customers that are in the pool.

To evaluate to what limit intensity of input traffic can be increased and conditions (4, 5) still be satisfied we can define input traffic as follows:

$$\Lambda = \sum_{k=1}^{d} \Lambda^k = \sum_{k=1}^{d} \Omega^k * \Theta^k . \tag{6}$$

Where Λ - is intensity of a customer traffic interring a system; Λ^k - is intensity of customer traffic, requiring *k-th* type of service; Ω^k - number of customers generating request for *k-th* type of services; Θ^k - intensity of requests for *k-th* type of service from each customer requesting this kind of service.

Proposed in [4] queuing system model allows define when conditions (4, 5) continue be satisfied upon changes in number of customers and variations in theirs requests for different kind of service. It also allows evaluate how failure of different serving links (serving elements) affects satisfaction of (4, 5).

However, stability in average does not define in full performance of a queuing system. Since number of customers in service is random, it is possible that at some moments of time $z^k(t)$ can exceed a predefined threshold α^k with probability P_{α^k}.

We can define a condition of a queuing system at moment *t* as *d*-mention vector

$$Z(t) = [z^1(t),..., z^k(t),..., z^d(t)] \qquad (7)$$

Let's also define Ξ as a set of system states *z(t)* on interval $t \in [T_1, T_2]$

$$\Xi = \{z^1(t) \le \alpha^1,..., z^k(t) \le \alpha^k,..., z^d(t) \le \alpha^d\} \qquad (8)$$

and $\alpha^k \succ M_{z^k}$ for all *k*.

When system is stable in average on interval $[T_1, T_2]$, we can use definition (3)

In the following analysis we will consider an open (with unlimited number of customers) queuing system. After initial stabilization this system is equifinal [5] and if for it

$\lim_{t \to \infty} M_{z^k}(t) = M_{z^k}$ then M_{z^k} does not depend on $z^k(0)$.

In analogy with [2] let use the following definitions:

1. Queuing system is stochastically stable in relationship to the triad $< \Pi, \Xi, \gamma >$ on interval $[T_1^{'} \ge T_1, T_2^{'} \le T_2]$ if from a fact that its parameters provide $M \in \Pi$ the following is correct:

$$P(Z(t) \notin \Xi \ \text{ for all } \ t \in [T_1^{'}, T_2^{'}]) \le \gamma \qquad (10)$$

2. Queuing system is stochastically stable in relationship to the triad $< \pi^k, \alpha^k \gamma^k >$ on interval $[T_1^{'} \ge T_1, T_2^{'} \le T_2]$ if from a fact that its parameters provide $M_{z^k} \le \pi^k$ the following is correct:

$$P(z^k(t) \geq a^k \text{ for } t \in [T_1^{'}, T_2^{'}]) \leq g^k \qquad (11)$$

In first case queuing system is stochastically stable as a whole; in second case it is only partially stochastically stable and only to some types of customers.

Let's consider second case. Values of a^k and g^k usually are defined by a system designer. However, it will be desirable to define value of p^k depending on behavior of $z^k(t)$. Since process of queuing system state changes is different for different systems, it is difficult to propose a general approach of defining p^k. However, if input and output flows of *k-th* customers can be consider Poisson and queuing system as open, then by using a^k and g^k , value of p^k can be defined via Martingale theory [6].

In corresponding with this theory a stochastic process is martingale if at moments of system state changes increments are the same and probabilities of such changes are equal. In case of considered now queuing system upon $\Lambda = const$ it will provide a condition:

$$P_{z^k+1} \approx P_{z^k} \qquad (12)$$

We also should mention that process $z^k(t)$ is continuing to the right for t and, as a result, is separable. Now, if $\{z(t), t \in [T_1 T_2]\}$ is a separable martingale, then, in general [8]:

$$P(\frac{\sup}{t \in [T_1 T_2]} z(t) \geq a) \leq \frac{M_z}{a} \qquad (13)$$

By using (13) and known a^k , g^k we can approximately define value of p^k. To satisfy condition (11) we have to provide $\frac{M_{z^k}}{a^k} \leq g^k$. As a result

$$p^k = M_{z^k} = a^k g^k \qquad (14)$$

If condition $M_{z^k} \leq p^k$ is satisfied for all k, then $M_z \in \Pi$, and $Z(t) \in \Xi$ with probability

$$\Psi = \prod_{k=1}^{d} (1 - g^{k}) \qquad (15)$$

Equation (15) defines system stochastic stability as a whole.

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